

Notes on the Riccati operator equation in open quantum systems

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A recent problem [B. Gardas, J. Math. Phys. **52**, 042104 (2011)] concerning an antilinear solution of the Riccati equation is solved. We also exemplify that a simplification of the Riccati equation, even under reasonable assumptions, can lead to a not equivalent equation.

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I. INTRODUCTION

Recently, the Riccati operator equation (RE) has attracted an attention to mathematical physicists (see *e.g.*, Refs.^{1,2}). As it has been argued, the solution of the equation can be used to obtain reduced time evolution ρ_t of a two-level open quantum system^{3,4}. This, so-called the reduced dynamics, plays a central role both in the quantum information theory and the theory of quantum information processing. It is well-established that $\rho_t = \Phi_t(\rho_0)$, where Φ_t is the trace preserving (TP) and completely positive (CP) map^{5,6}. Such TP-CP maps, known as channels, are commonly used to simulate real quantum systems and to design quantum algorithms^{7,8}.

To understand a connection between a given TP-CP map Φ_t and a solution of the RE, let us begin by reviewing some basics facts concerning open quantum systems. For this purpose we consider the system-environment Hamiltonian in the following form

$$\mathbf{H} = H_Q \otimes \mathbb{I}_E + \mathbb{I}_Q \otimes H_E + \mathbf{H}_{\text{int}}, \quad (1)$$

where H_Q and H_E represent the Hamiltonian of the qubit and the environment, respectively. \mathbf{H}_{int} specifies the interaction between the systems, whereas \mathbb{I}_Q and \mathbb{I}_E are the identity operators on corresponding Hilbert spaces \mathbb{C}^2 and \mathcal{H}_E of the qubit and its environment, respectively. \mathbf{H} acts on the Hilbert space $\mathcal{H}_{\text{tot}} = \mathbb{C}^2 \otimes \mathcal{H}_E$ and it admits the block operator matrix representation⁹:

$$\mathbf{H} = \begin{bmatrix} H_+ & V \\ V^\dagger & H_- \end{bmatrix} \quad \text{on} \quad \mathcal{D}(\mathbf{H}) = \mathcal{D}(H_+) \oplus \mathcal{D}(H_-), \quad (2)$$

with respect to the decomposition $\mathcal{H}_{\text{tot}} = \mathcal{H}_E \oplus \mathcal{H}_E$. The diagonal entries $H_\pm : \mathcal{D}(H_\pm) \rightarrow \mathcal{H}_E$ are densely defined and self-adjoint unbounded operators. We have assumed that $\mathcal{D}(V) = \mathcal{D}(V^\dagger) = \mathcal{H}_E$, which means that V , and thus V^\dagger as well, is bounded.

The time reduced evolution of an open system is given by

$$\Phi_t(\rho_0) = \text{Tr}_E[\mathbf{U}_t \Psi(\rho_0) \mathbf{U}_t^\dagger], \quad (3)$$

where $\mathbf{U}_t = \exp(-i\mathbf{H}t)$ is the time evolution operator of the total system. The map Ψ assigns to each initial state ρ_0 a single state $\Psi(\rho_0)$ of the total system. This assignment map must be chosen properly so that Φ_t can be well-defined¹⁰. In this paper we assume that no

correlations between the systems are initially present¹¹, and thus we take $\Psi(\rho_0) = \rho_0 \otimes \omega$, for some initial state of the environment ω .

The linear map Tr_E denotes the so-called partial trace:

$$\text{Tr}_E \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} = \begin{pmatrix} \text{Tr} M_{11} & \text{Tr} M_{12} \\ \text{Tr} M_{21} & \text{Tr} M_{22} \end{pmatrix} \in M_2(\mathbb{C}), \quad \text{where } M_{ij} \in \mathcal{T}(\mathcal{H}_E). \quad (4)$$

In the above description, Tr refers to the usual trace on \mathcal{H}_E , $\mathcal{T}(\mathcal{H}_E)$ denotes the Banach space of trace class operators with the trace norm: $\|A\|_1 = \text{Tr}(\sqrt{AA^\dagger})$, whereas $M_2(\mathbb{C})$ is the Banach space of 2×2 complex matrices. From (4) one can see that the partial trace in Eq. (3) transforms states of the total system (block operator matrices, square bracket) into states of the open system (complex matrices, round bracket). It is worth noting that the partial trace Tr_E , unlike the trace Tr , is not cyclic, namely $\text{Tr}_E(\mathbf{A}\mathbf{B}) \neq \text{Tr}_E(\mathbf{B}\mathbf{A})$.

In general, the formula (3) is far less useful, than its simplicity might indicate. Indeed, to trace out the state $\mathbf{U}_t \Psi(\rho_0) \mathbf{U}_t^\dagger$ over the environment degrees of freedom one needs to find a decomposition of \mathbf{U}_t with respect to the decomposition of Hilbert space $\mathcal{H}_{\text{tot}} = \mathcal{H}_E \oplus \mathcal{H}_S$. This problem is fundamental in quantum mechanics, as it requires the diagonalization of \mathbf{H} .

It is well-established (see *e.g.*¹² and Refs. therein) that \mathbf{H} can be block-diagonalized provided there exist a bounded solution X (with $\text{Ran}(X)|_{\mathcal{D}(H_+)} \subset \mathcal{D}(H_-)$) of the Riccati equation:

$$XVX + XH_+ - H_-X - V^\dagger = 0 \quad \text{on } \mathcal{D}(H_+). \quad (5)$$

To be more specific, if X solves (5) the following equality holds true:

$$\mathbf{S}^{-1} \mathbf{H} \mathbf{S} = \begin{bmatrix} Z_+ & 0 \\ 0 & Z_- \end{bmatrix}, \quad \text{where } \mathbf{S} = \begin{bmatrix} \mathbb{I}_E & -X^\dagger \\ X & \mathbb{I}_E \end{bmatrix} \quad (6)$$

and $Z_+ = H_+ + VX$, $Z_- = H_- - V^\dagger X^\dagger$ with $\mathcal{D}(Z_\pm) = \mathcal{D}(H_\pm)$. Moreover \mathbf{S}^{-1} exists as a bounded operator².

Using decomposition (6) one can write \mathbf{U}_t in an explicit matrix form:

$$\mathbf{U}_t = \exp \left(-i \mathbf{S} \begin{bmatrix} Z_+ & 0 \\ 0 & Z_- \end{bmatrix} \mathbf{S}^{-1} t \right) = \mathbf{S} \begin{bmatrix} e^{-iZ_+t} & 0 \\ 0 & e^{-iZ_-t} \end{bmatrix} \mathbf{S}^{-1}. \quad (7)$$

Among all difficulties concerning the RE, we will focus on two problems described below. The first one¹ arises when the solution of the RE is antilinear. In this case, the formula (7)

cannot be applied, since \mathbf{S} is neither an antilinear nor a linear operator. Therefore, we are interested in asking how can we determine the time evolution of the total system.

The RE is an operator equation and even for a simple system it may be problematic to find the solution. However, there are cases in which the RE can be simplified considerably¹. For instance, in the case when $V = \alpha \mathbb{I}_E \equiv \alpha$, where $\alpha \in \mathbb{R}$; then the RE reads

$$\alpha X^2 + XH_+ - H_-X - \alpha = 0 \quad \text{on} \quad \mathcal{D}(H_+). \quad (8)$$

In the instance when the operators H_{\pm} are self-adjoint and commute (*e.g.* $[H_+, H_-] = 0$), it is reasonable to assume, that the solution has the following form $X = f(H_+)$ or $X = g(H_-)$. Then, the functions f and g read $f = h|_{\sigma(H_+)}$ and $g = h|_{\sigma(H_-)}$, where h satisfies the quadratic equation: $\alpha h(\lambda)^2 + 2h(\lambda)\lambda - \alpha = 0$. At this point the second issue can be addressed: does every solution of (8) is an analytical function of H_+ or H_- ?

The primary goal of the presented work is to solve this two worthwhile issues. First, we show that there is a simple way of computing the evolution operator \mathbf{U}_t with the use of a formula similar to the one provided by the Eq. (7). Next, we exemplify that not every solution of (8) is an analytical function of H_+ or H_- .

II. ANTILINEAR SOLUTION

In this section we derive a simple method of computing \mathbf{U}_t in the case when the solution of the RE is antilinear. To end this let us assume that τ is a bounded, antilinear (*i.e.* $\tau(z|\psi\rangle) = z^*\tau|\psi\rangle$) operator which solves Eq. (5) and let $\mathbf{S}_{\tau} = \frac{1}{\sqrt{2}}\mathbf{S}$, where \mathbf{S} denotes matrix from Eq. (6) with $X = \tau$. Although, \mathbf{S}_{τ} is neither a linear nor an antilinear in this case, it has an interesting property:

$$\mathbf{S}_{\tau}(z|\psi\rangle) = \Re(z)\mathbf{S}_{\tau}|\psi\rangle + i\Im(z)\mathbf{S}_{-\tau}|\psi\rangle \quad (\text{note} \quad \mathbf{S}_{-\tau}\mathbf{S}_{\tau} = \mathbb{I}_E = \mathbf{S}_{\tau}\mathbf{S}_{-\tau}), \quad (9)$$

which reduces to $\mathbf{S}_{\tau}(z|\psi\rangle) = z\mathbf{S}_{\tau}|\psi\rangle$, for $z \in \mathbb{R}$. Therefore, \mathbf{S}_{τ} can be treated as a linear operator when it acts on some subspace $\mathcal{H}_{\mathbb{R}}$ over a field of the *real* numbers \mathbb{R} . In other words, the restricted operator $\mathbf{S}_{\tau|_{\mathcal{H}_{\mathbb{R}}}}$ is linear.

As a result, if $\mathbf{H} = \mathbf{S}_{\tau}\mathbf{H}_d\mathbf{S}_{-\tau}$ then for an analytical function $f : \sigma(\mathbf{H}) \rightarrow \mathbb{R}$ we have $f(\mathbf{S}_{\tau}\mathbf{H}_d\mathbf{S}_{-\tau}) = \mathbf{S}_{\tau}f(\mathbf{H}_d)\mathbf{S}_{-\tau}$, thus in particular we obtain

$$\exp(-i\mathbf{H}t) = \mathbf{S}_{\tau} \cos(\mathbf{H}_d t) \mathbf{S}_{-\tau} - i\mathbf{S}_{\tau} \sin(\mathbf{H}_d t) \mathbf{S}_{-\tau}. \quad (10)$$

Although the right side of the Eq. (10) is *not* equal to $\mathbf{S}_\tau \exp(-i\mathbf{H}_d t) \mathbf{S}_{-\tau}$, yet the evolution operator can still easily be managed by separately transforming $\cos(\mathbf{H}t)$ and $\sin(\mathbf{H}t)$ functions, and then combining the results by using the Euler formula.

III. NON-ANALYTICAL SOLUTION OF THE RICCATI EQUATION

In this section we give an example that the solution of (8) is not necessarily an analytical function of H_+ or H_- . To achieve this, we consider the RE (8) with $H_\pm = \pm H$, where $H : \mathcal{D}(H) \rightarrow \mathcal{H}_E$ is assumed to be densely defined and self-adjoint operator. With the use of these assumptions the RE on $\mathcal{D}(H)$ can be written as

$$\alpha X^2 + XH + HX - \alpha = 0. \quad (11)$$

If one assumes that the solution is in the form $X = f(H)$, then the RE (11) simplifies on $\mathcal{D}(H)$ to the quadratic operator equation:

$$\alpha X^2 + 2HX - \alpha = 0. \quad (12)$$

It is not difficult to see that the function f is a solution of the quadratic equation:

$$\alpha f(\lambda)^2 + 2\lambda f(\lambda) - \alpha = 0, \quad \text{where } \lambda \in \sigma(H). \quad (13)$$

Since there is only one self-adjoint operator in Eq. (11), the assumption $X = f(H)$ seems to be reasonable. The question concerning analytical solutions of RE in this case reduces to the question: are the Eqs. (11) and (12) equivalent? We will exemplify that the answer in general is negative, and we will prove it choosing H such that the solution of (11) does not solve (12).

We construct H as follows. Let a^\dagger and a be the bosonic creation and annihilation operators, respectively. They are defined on a common domain \mathcal{D}_1 and obey the canonical commutation relation: $[a, a^\dagger] = \mathbb{I}$, on some dense subset \mathcal{D}_2 of \mathcal{D}_1 . In terms of these operators we define the number operator $N = a^\dagger a$, which is self-adjoint on \mathcal{D}_2 . An explicit form of the sets \mathcal{D}_k is given by¹³

$$\mathcal{D}_k = \left\{ |\psi\rangle : \sum_{n=0}^{\infty} n^k |\langle \phi_n | \psi \rangle|^2 < \infty \right\}, \quad k = 1, 2; \quad (14)$$

where the vectors $|\phi_n\rangle \in \mathcal{D}_2$ satisfy $N|\phi_n\rangle = n|\phi_n\rangle$. Moreover, on \mathcal{D}_1 the creation and annihilation operators can be defined explicitly as

$$a|\phi\rangle = \sum_{n=1}^{\infty} \sqrt{n} \langle \phi_n | \phi \rangle |\phi_{n-1}\rangle, \quad a^\dagger|\phi\rangle = \sum_{n=0}^{\infty} \sqrt{n+1} \langle \phi_n | \phi \rangle |\phi_{n+1}\rangle, \quad |\phi\rangle \in \mathcal{D}_1. \quad (15)$$

Finally, we take $H = g^*a + ga^\dagger$ with $\mathcal{D}(H) = \mathcal{D}_1$, where $g \in \mathbb{C}$.

It is not difficult to see that the solution of the RE (11) is given by the well-known bosonic parity operator¹⁴:

$$P|\psi\rangle = \sum_{n=0}^{\infty} e^{i\pi n} \langle \phi_n | \psi \rangle |\phi_n\rangle, \quad |\psi\rangle \in \mathcal{H}_E. \quad (16)$$

Indeed, from (16) follows that P is both Hermitian and unitary, in particular $P^2 = \mathbb{I}_E$. Therefore, in order to prove that P solves (11), it is sufficient to show that $PHP = -H$. The latter is obvious since $PaP = -a$ and $Pa^\dagger P = -a^\dagger$.

Note, formally P can be written as $P = \exp(i\pi N)$, however, unlike N it is everywhere defined, thus it is bounded as well ($\|P\| = 1$). Of course, P is not an analytical function of H and it does not solve Eq. (12) because $PH \neq 0$.

IV. SUMMARY

We have proposed a simple resolution of the recent problem concerning antilinear solution of the Riccati equation. Our result allows to take advantage of the result (7), in the case of an antilinear decomposition of the total Hamiltonian. The solution given in this paper is not as complicate as it is crafty. It relays on the very same exponential formula that is uses for a linear decomposition of the total Hamiltonian. However, the way it is used is different as we have explained in Sec. II.

We have also argued that a simplification of the RE, even under reasonable assumptions, can lead to the not equivalent equation.

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REFERENCES

- ¹B. Gardas, *J. Math. Phys.* **51**, 062103 (2010); *J. Math. Phys.* **52**, 042104 (2011); *Phys. Rev. A* **82**, 042115 (2010); *J. Phys. A: Math. Theor.* **44**, 195301 (2011).
- ²B. Gardas and Z. Puchała, *J. Phys. A: Math. Theor.* **44**, 215306 (2011).
- ³H. P. Breuer and F. Petruccione, *The Theory of Open Quantum Systems* (Oxford University Press, Oxford, 2002).
- ⁴E. B. Davies, *Quantum theory of open systems* (Academic Press, London, 1976).
- ⁵M.-D. Choi, *Linear Algebra and its Applications* **10**, 285 (1975).
- ⁶V. Gorini, A. Kossakowski, and E. C. G. Sudarshan, *J. Math. Phys.* **17**, 821 (1976).
- ⁷M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, 2000).
- ⁸M. Hayashi, *Quantum Information: An Introduction* (Springer-Verlag, Berlin, 2006).
- ⁹C. Tretter, *Spectral Theory of Block Operator Matrices and Applications* (Imperial College Press, London, 2008).
- ¹⁰P. Pechukas, *Phys. Rev. Lett.* **73**, 1060 (1994); *Phys. Rev. Lett.* **75**, 3021 (1995); R. Al-icki, *Phys. Rev. Lett.* **75**, 3020 (1995).
- ¹¹P. Štelmachovič and V. Bužek, *Phys. Rev. A* **64**, 062106 (2001); *Phys. Rev. A* **67**, 029902 (2003).
- ¹²V. Kostykin, K. A. Makarov, and A. K. Motovilov, *Contemporary Mathematics* **327**, 181 (2003).
- ¹³F. H. Szafraniec, *Contemp. Math.* **212**, 269 (1998); *Rep. Math. Phys.* **59**, 401 (2007); *J. Phys. A: Math. and Gen.* **34**, 10487 (2001).
- ¹⁴C. M. Bender, P. N. Meisinger, and Q. Wang, *J. Phys. A: Math. Gen.* **36**, 1029 (2003).